

MIRROR DUALITY VIA G_2 AND $\text{Spin}(7)$ MANIFOLDS

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ABSTRACT. The main purpose of this paper is to give a construction of certain “mirror dual” Calabi-Yau submanifolds inside of a G_2 manifold. More specifically, we explain how to assign a G_2 manifold (M, φ, Λ) , with the calibration 3-form φ and an oriented 2-plane field Λ , a pair of parametrized tangent bundle valued 2 and 3-forms of M . These forms can then be used to define different complex and symplectic structures on certain 6-dimensional subbundles of $T(M)$. When these bundles are integrated they give mirror CY manifolds. In a similar way, one can define mirror dual G_2 manifolds inside of a $\text{Spin}(7)$ manifold (N^8, Ψ) . In case N^8 admits an oriented 3-plane field, by iterating this process we obtain Calabi-Yau submanifold pairs in N whose complex and symplectic structures determine each other via the calibration form of the ambient G_2 (or $\text{Spin}(7)$) manifold.

1. INTRODUCTION

Let (M^7, φ) be a G_2 manifold with the calibration 3-form φ . If φ restricts to be the volume form of an oriented 3-dimensional submanifold Y^3 , then Y is called an associative submanifold of M . Associative submanifolds are very interesting objects as they behave very similarly to holomorphic curves of Calabi-Yau manifolds.

In [AS], we studied the deformations of associative submanifolds of (M, φ) in order to construct Gromov-Witten like invariants. One of our main observations was that oriented 2-plane fields on M always exist by a theorem of Thomas [T], and by using them one can split the tangent bundle $T(M) = \mathbf{E} \oplus \mathbf{V}$ as an orthogonal direct sum of an associative 3-plane bundle \mathbf{E} and a complex 4-plane bundle \mathbf{V} . This allows us to define “complex associative submanifolds” of M , whose deformation equations may be reduced to the Seiberg-Witten equations, and hence we can assign local invariants to them, and assign various invariants to (M, φ, Λ) , where Λ is an oriented 2-plane field on M . It turns out that these Seiberg-Witten equations on the submanifolds are restrictions of global equations on M .

In this paper, we explain how the geometric structures on G_2 manifolds with oriented 2-plane fields (M, φ, Λ) provide complex and symplectic structures to certain 6-dimensional subbundles of $T(M)$. When these bundles integrated we obtain a pair of Calabi-Yau manifolds whose complex and symplectic structures are remarkably related to each other. We also study examples of Calabi-Yau manifolds which fit

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nicely in our mirror set-up. Later, we do similar constructions for $Spin(7)$ manifolds with oriented 3-plane fields. We then explain how these structures lead to the definition of “dual G_2 manifolds” in a $Spin(7)$ manifold, with their own dual Calabi-Yau submanifolds.

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2. ASSOCIATIVE AND COMPLEX DISTRIBUTIONS OF A G_2 MANIFOLD

Let us go through quickly over the basic definitions about G_2 manifolds. The main references are the two foundational papers [HL] and [B1], as well as [S], [B2], [BS], and [J]. We also need some properties introduced in [AS]. Now let $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ be the octonions which is an 8 dimensional division algebra generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$, and let $im\mathbb{O} = \mathbb{R}^7$ be the imaginary octonions with the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \rightarrow \mathbb{R}^7$, defined by $u \times v = im(\bar{v}.u)$. The exceptional Lie group G_2 is the linear automorphisms of $im\mathbb{O}$ preserving this cross product operation, it can also be defined in terms of the orthogonal 3-frames in \mathbb{R}^7 :

$$G_2 = \{(u_1, u_2, u_3) \in (im\mathbb{O})^3 \mid \langle u_i, u_j \rangle = \delta_{ij}, \langle u_1 \times u_2, u_3 \rangle = 0\}.$$

Another very useful definition popularized in [B1] is the subgroup of $GL(7, \mathbb{R})$ which fixes a particular 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$. Denote $e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)$, then

$$G_2 = \{A \in GL(7, \mathbb{R}) \mid A^*\varphi_0 = \varphi_0\}.$$

$$(1) \quad \varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

Definition 1. A smooth 7-manifold M^7 has a G_2 structure if its tangent frame bundle reduces to a G_2 bundle. Equivalently, M^7 has a G_2 structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$ (pointwise condition). We call (M, φ) a manifold with G_2 structure.

A G_2 structure φ on M^7 gives an orientation $\mu \in \Omega^7(M)$ on M , and μ determines a metric $g = g_\varphi = \langle \cdot, \cdot \rangle$ on M , and a cross product structure \times on the tangent bundle of M as follows: Let $i_v = v \lrcorner$ be the interior product with a vector v , then

$$(2) \quad \langle u, v \rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi] / 6\mu.$$

$$(3) \quad \varphi(u, v, w) = \langle u \times v, w \rangle.$$

Definition 2. A manifold with G_2 structure (M, φ) is called a G_2 manifold if the holonomy group of the Levi-Civita connection (of the metric g_φ) lies inside of G_2 . Equivalently (M, φ) is a G_2 manifold if φ is parallel with respect to the metric g_φ , that is $\nabla_{g_\varphi}(\varphi) = 0$; which is equivalent to $d\varphi = 0$, $d(*_{g_\varphi}\varphi) = 0$. Also equivalently, at each point $x_0 \in M$ there is a chart $(U, x_0) \rightarrow (\mathbb{R}^7, 0)$ on which φ equals to φ_0 up to second order term, i.e. on the image of U $\varphi(x) = \varphi_0 + O(|x|^2)$.

Remark 1. One important class of G_2 manifolds are the ones obtained from Calabi-Yau manifolds. Let (X, ω, Ω) be a complex 3-dimensional Calabi-Yau manifold with Kähler form ω and a nowhere vanishing holomorphic 3-form Ω , then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a G_2 manifold. In this case $\varphi = \text{Re } \Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact G_2 manifold.

Definition 3. Let (M, φ) be a G_2 manifold. A 4-dimensional submanifold $X \subset M$ is called coassociative if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called associative if $\varphi|_Y \equiv \text{vol}(Y)$; this condition is equivalent to the condition $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$(4) \quad \langle \chi(u, v, w), z \rangle = * \varphi(u, v, w, z)$$

The equivalence of these conditions follows from the ‘associator equality’ of [HL]

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2$$

Similar to the definition of χ one can define a tangent bundle 2-form, which is just the cross product of M (nevertheless viewing it as a 2-form has its advantages).

Definition 4. Let (M, φ) be a G_2 manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle valued 2-form defined by the identity:

$$(5) \quad \langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle$$

Now we have two useful properties from [AS], the first property basically follows from definitions, the second property fortunately applies when the first property fails to give anything useful.

Lemma 1. ([AS]) To any 3-dimensional submanifold $Y^3 \subset (M, \varphi)$, χ assigns a normal vector field, which vanishes when Y is associative.

Lemma 2. ([AS]) To any associative manifold $Y^3 \subset (M, \varphi)$ with a non-vanishing oriented 2-plane field, χ defines a complex structure on its normal bundle (notice in particular that any coassociative submanifold $X \subset M$ has an almost complex structure if its normal bundle has a non-vanishing section).

Proof. Let $L \subset \mathbb{R}^7$ be an associative 3-plane, that is $\varphi_0|_L = \text{vol}(L)$. Then for every pair of orthonormal vectors $\{u, v\} \subset L$, the form χ defines a complex structure on the orthogonal 4-plane L^\perp , as follows: Define $j : L^\perp \rightarrow L^\perp$ by

$$(6) \quad j(X) = \chi(u, v, X)$$

This is well defined i.e. $j(X) \in L^\perp$, because when $w \in L$ we have:

$$\langle \chi(u, v, X), w \rangle = * \varphi_0(u, v, X, w) = - * \varphi_0(u, v, w, X) = \langle \chi(u, v, w), X \rangle = 0$$

Also $j^2(X) = j(\chi(u, v, X)) = \chi(u, v, \chi(u, v, X)) = -X$. We can check the last equality by taking an orthonormal basis $\{X_j\} \subset L^\perp$ and calculating

$$\begin{aligned} \langle \chi(u, v, \chi(u, v, X_i)), X_j \rangle &= * \varphi_0(u, v, \chi(u, v, X_i), X_j) = - * \varphi_0(u, v, X_j, \chi(u, v, X_i)) \\ &= - \langle \chi(u, v, X_j), \chi(u, v, X_i) \rangle = -\delta_{ij} \end{aligned}$$

The last equality holds since the map j is orthogonal, and the orthogonality can be seen by polarizing the associator equality, and by noticing $\varphi_0(u, v, X_i) = 0$. Observe that the map j only depends on the oriented 2-plane $\Lambda = \langle u, v \rangle$ generated by $\{u, v\}$ (i.e. it only depends on the complex structure on Λ). \square

Remark 2. Notice that Lemma 1 gives an interesting flow on the 3-dimensional submanifolds of G_2 manifolds $f : Y \hookrightarrow (M, \varphi)$ (call χ -flow), described by:

$$\frac{\partial}{\partial t} f = \chi(f_* \text{vol}(Y))$$

For example, by [BS] the total space of the spinor bundle $Q^7 \rightarrow S^3$ (with \mathbb{C}^2 fibers) is a G_2 manifold, and the zero section $S^3 \subset Q$ is an associative submanifold. We can imbed any homotopy 3-sphere Σ^3 into Q (homotopic to the zero-section). We conjecture that χ -flow on $\Sigma \subset Q$, takes Σ diffeomorphically onto the zero section. Note that, since any S^3 smoothly unknots in S^7 it is not possible to produce quick counterexamples by tying local knots; and an affirmative answer gives $\Sigma \cong S^3$.

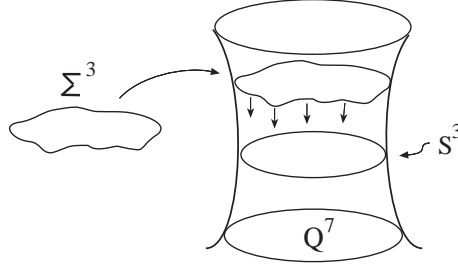


FIGURE 1.

Finally, we need some identities from [B2] (also see [K]) for (M^7, φ) , which follow from local calculations by using the definition (1). For $\beta \in \Omega^1(M)$ we have:

$$(7) \quad |\varphi \wedge \beta|^2 = 4|\beta|^2, \text{ and } |*\varphi \wedge \beta|^2 = 3|\beta|^2,$$

$$(8) \quad (\xi \lrcorner \varphi) \wedge \varphi = 2 * (\xi \lrcorner \varphi), \text{ and } * [*(\beta \wedge *\varphi) \wedge *\varphi] = 3\beta,$$

$$(9) \quad \beta \times (\beta \times u) = -|\beta|^2 u + \langle \beta, u \rangle \beta,$$

where $*$ is the star operator. Let ξ be a vector field on any Riemannian manifold (M, g) , and $\xi^\# \in \Omega^1(M)$ be its dual 1-form, i.e. $\xi^\#(v) = \langle \xi, v \rangle$. Then for $\alpha \in \Omega^k(M)$:

$$(10) \quad *(\xi \lrcorner \alpha) = (-1)^{k+1}(\xi^\# \wedge * \alpha).$$

3. MIRROR DUALITY IN G_2 MANIFOLDS

On a local chart of a G_2 manifold (M, φ) , the form φ coincides with the form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ up to quadratic terms, we can express the corresponding tangent valued forms χ and ψ in terms of φ_0 in local coordinates. More generally, if e_1, \dots, e_7 is any local orthonormal frame and e^1, \dots, e^7 is the dual frame, from definitions we get:

$$\begin{aligned} \chi = & (e^{256} + e^{247} + e^{346} - e^{357})e_1 \\ & + (-e^{156} - e^{147} - e^{345} - e^{367})e_2 \\ & + (e^{157} - e^{146} + e^{245} + e^{267})e_3 \\ & + (e^{127} + e^{136} - e^{235} - e^{567})e_4 \\ & + (e^{126} - e^{137} + e^{234} + e^{467})e_5 \\ & + (-e^{125} - e^{134} - e^{237} - e^{457})e_6 \\ & + (-e^{124} + e^{135} + e^{236} + e^{456})e_7. \end{aligned}$$

$$\begin{aligned} \psi = & (e^{23} + e^{45} + e^{67})e_1 \\ & + (e^{46} - e^{57} - e^{13})e_2 \\ & + (e^{12} - e^{47} - e^{56})e_3 \\ & + (e^{37} - e^{15} - e^{26})e_4 \\ & + (e^{14} + e^{27} + e^{36})e_5 \\ & + (e^{24} - e^{17} - e^{35})e_6 \\ & + (e^{16} - e^{25} - e^{34})e_7. \end{aligned}$$

The forms χ and ψ induce complex and symplectic structures on certain subbundles of $T(M)$ as follows: Let ξ be a nonvanishing vector field of M . We can define a symplectic ω_ξ and a complex structure J_ξ on the 6-plane bundle $V_\xi := \xi^\perp$ by

$$(11) \quad \omega_\xi = \langle \psi, \xi \rangle \quad \text{and} \quad J_\xi(X) = X \times \xi.$$

Now we can define

$$(12) \quad \text{Re } \Omega_\xi = \varphi|_{V_\xi} \quad \text{and} \quad \text{Im } \Omega_\xi = \langle \chi, \xi \rangle.$$

In particular $\omega_\xi = \xi \lrcorner \varphi$, and $\text{Im } \Omega_\xi = \xi \lrcorner * \varphi$. Call $\Omega_\xi = \text{Re } \Omega_\xi + i \text{Im } \Omega_\xi$. The reason for defining these is to pin down a Calabi-Yau like structure on any G_2 manifold. In case $(M, \varphi) = CY \times S^1$ these quantities are related to the ones in Remark 1. Notice that when $\xi \in \mathbf{E}$ then J_ξ is an extension of J of Lemma 2 from the 4-dimensional bundle \mathbf{V} to the 6-dimensional bundle V_ξ .

By choosing different directions, i.e. different ξ , one can find the corresponding complex and symplectic structures. In particular we will get two different complex structures if we choose ξ in the associative subbundle \mathbf{E} (where φ restricts to be 1), or if we choose ξ in the complementary subbundle \mathbf{V} , which we will call the coassociative subbundle. Note that φ restricts to zero on the coassociative subbundle.

In local coordinates, it is a straightforward calculation that by choosing $\xi = e_i$ for any i , from equations (11) and (12), we can easily obtain the corresponding structures ω_ξ , J_ξ , Ω_ξ . For example, let us assume that $\{e_1, e_2, e_3\}$ is the local orthonormal basis for the associative bundle \mathbf{E} , and $\{e_4, e_5, e_6, e_7\}$ is the local orthonormal basis for the coassociative bundle \mathbf{V} . Then if we choose $\xi = e_3 = e_1 \times e_2$ then we get $\omega_\xi = e^{12} - e^{47} - e^{56}$ and $\text{Im } \Omega_\xi = e^{157} - e^{146} + e^{245} + e^{267}$. On the other hand, if we choose $\xi = e_7$ then $\omega_\xi = e^{16} - e^{25} - e^{34}$ and $\text{Im } \Omega_\xi = -e^{124} + e^{135} + e^{236} + e^{456}$ which will give various symplectic and complex structures on the bundle V_ξ .

3.1. A useful example.

Let us take a Calabi-Yau 6-torus $\mathbb{T}^6 = \mathbb{T}^3 \times \mathbb{T}^3$, where $\{e_1, e_2, e_3\}$ is the basis for one \mathbb{T}^3 and $\{e_4, e_5, e_6\}$ the basis for the other (terms expressed with a slight abuse of notation). We can take the product $M = \mathbb{T}^6 \times S^1$ as the corresponding G_2 manifold with the calibration 3-form $\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$, and with the decomposition $T(M) = \mathbf{E} \oplus \mathbf{V}$, where $\mathbf{E} = \{e_1, e_2, e_3\}$ and $\mathbf{V} = \{e_4, e_5, e_6, e_7\}$. Now, if we choose $\xi = e_7$, then $V_\xi = \langle e_1, \dots, e_6 \rangle$ and the symplectic form is $\omega_\xi = e^{16} - e^{25} - e^{34}$, and the complex structure is

$$J_\xi = \begin{pmatrix} \mathbf{e}_1 & \mapsto & -e_6 \\ \mathbf{e}_2 & \mapsto & e_5 \\ \mathbf{e}_3 & \mapsto & e_4 \end{pmatrix}$$

and the complex valued $(3, 0)$ form is $\Omega_\xi = (e^1 + ie^6) \wedge (e^2 - ie^5) \wedge (e^3 - ie^4)$; note that this is just $\Omega_\xi = (e^1 - iJ_\xi(e^1)) \wedge (e^2 - iJ_\xi(e^2)) \wedge (e^3 - iJ_\xi(e^3))$.

On the other hand, if we choose $\xi' = e_3$ then $V_{\xi'} = \langle e_1, \dots, \hat{e}_3, \dots, e_7 \rangle$ and the symplectic form is $\omega_{\xi'} = e^{12} - e^{47} - e^{56}$ and the complex structure is

$$J_{\xi'} = \begin{pmatrix} \mathbf{e}_1 & \mapsto & -\mathbf{e}_2 \\ e_4 & \mapsto & e_7 \\ e_5 & \mapsto & e_6 \end{pmatrix}$$

Also $\Omega_{\xi'} = (e^1 + ie^2) \wedge (e^4 - ie^7) \wedge (e^5 - ie^6)$, as above this can be expressed more tidily as $\Omega_{\xi'} = (e^1 - iJ_{\xi'}(e^1)) \wedge (e^4 - iJ_{\xi'}(e^4)) \wedge (e^5 - iJ_{\xi'}(e^5))$. In the expressions of J 's the basis of associative bundle \mathbf{E} is indicated by bold face letters to indicate the differing complex structures on \mathbb{T}^6 . To sum up: If we choose ξ from the coassociative bundle \mathbf{V} we get the complex structure which decomposes the 6-torus as $\mathbb{T}^3 \times \mathbb{T}^3$. On the other hand if we choose ξ from the associative bundle \mathbf{E} then the induced complex structure on the 6-torus corresponds to the decomposition as $\mathbb{T}^2 \times \mathbb{T}^4$. This is the phenomenon known as “mirror duality”. Here these two $SU(3)$ and $SU(2)$ structures are different but they come from the same φ hence they are dual. These examples suggests the following definition of “mirror duality” in G_2 manifolds:

Definition 5. *Two Calabi-Yau manifolds are mirror pairs of each other, if their complex structures are induced from the same calibration 3-form in a G_2 manifold. Furthermore we call them strong mirror pairs if their normal vector fields ξ and ξ' are homotopic to each other through nonvanishing vector fields.*

Remark 3. *In the above example of $CY \times S^1$, where $CY = \mathbb{T}^6$, the calibration form $\varphi = \text{Re } \Omega + \omega \wedge dt$ gives Lagrangian tori fibration in X_ξ and complex tori fibration in $X_{\xi'}$. They are different manifestations of φ residing on one higher dimensional G_2 manifold M^7 . In the next section this correspondence will be made precise.*

In Section 4.2 we will discuss a more general notion of mirror Calabi-Yau manifold pairs, when they sit in different G_2 manifolds, which are themselves mirror duals of each other in a $Spin(7)$ manifold.

3.2. General setting.

Let (M^7, φ, Λ) be a manifold with a G_2 structure and a non-vanishing oriented 2-plane field. As suggested in [AS] we can view (M^7, φ) as an analog of a symplectic manifold, and the 2-plane field Λ as an analog of a complex structure taming φ . This is because Λ along with φ gives the associative/complex bundle splitting $T(M) = \mathbf{E}_{\varphi, \Lambda} \oplus \mathbf{V}_{\varphi, \Lambda}$. Now, the next object is a choice of a non-vanishing unit vector field $\xi \in \Omega^0(M, TM)$, which gives a codimension one distribution $V_\xi := \xi^\perp$ on M , which is equipped with the structures $(V_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ as given by (11) and (12).

Let $\xi^\#$ be the dual 1-form of ξ . Let $e_{\xi^\#}$ and $i_\xi = \xi_\perp$ denote the exterior and interior product operations on differential forms. Clearly $e_{\xi^\#} \circ i_\xi + i_\xi \circ e_{\xi^\#} = \text{id}$.

$$(13) \quad \varphi = e_{\xi^\#} \circ i_\xi(\varphi) + i_\xi \circ e_{\xi^\#}(\varphi) = \omega_\xi \wedge \xi^\# + \text{Re } \Omega_\xi.$$

This is just the decomposing of the form φ with respect to $\xi \oplus \xi^\perp$. Recall that the condition that the distribution V_ξ be integrable (the involutive condition which implies ξ^\perp comes from a foliation) is given by:

$$(14) \quad d\xi^\# \wedge \xi^\# = 0.$$

Even when V_ξ is not integrable, by [Th] it is homotopic to a foliation. Assume X_ξ be a page of this foliation; for simplicity assume this 6-dimensional manifold is smooth.

It is clear from definitions that J_ξ is an almost complex structure on X_ξ . Also the 2-form ω_ξ is non-degenerate on X_ξ , because from (2) we can write

$$(15) \quad \omega_\xi^3 = (\xi \lrcorner \varphi)^3 = \xi \lrcorner [(\xi \lrcorner \varphi) \wedge (\xi \lrcorner \varphi) \wedge \varphi] = \xi \lrcorner (6|\xi|^2 \mu) = 6\mu_\xi$$

where $\mu_\xi = \mu|_{V_\xi}$ is the induced orientation form on V_ξ .

Lemma 3. J_ξ is compatible with ω_ξ , and it is metric invariant.

Proof. Let $u, v \in V_\xi$

$$\begin{aligned} \omega_\xi(J_\xi(u), v) &= \omega_\xi(u \times \xi, v) = \langle \psi(u \times \xi, v), \xi \rangle = \varphi(u \times \xi, v, \xi) && \text{by (5)} \\ &= -\varphi(\xi, \xi \times u, v) = -\langle \xi \times (\xi \times u), v \rangle && \text{by (3)} \\ &= -\langle -|\xi|^2 u + \langle \xi, u \rangle \xi, v \rangle = |\xi|^2 \langle u, v \rangle - \langle \xi, u \rangle \langle \xi, v \rangle && \text{by (9)} \\ &= \langle u, v \rangle. \end{aligned}$$

By plugging in $J_\xi(u), J_\xi(v)$ for u, v : $\langle J_\xi(u), J_\xi(v) \rangle = -\omega_\xi(u, J_\xi(v)) = \langle u, v \rangle$ \square

Lemma 4. Ω_ξ is a non-vanishing $(3, 0)$ form.

Proof. By a local calculation as in Section 3.1 we see that Ω_ξ is a $(3, 0)$ form, and is non-vanishing because $\Omega_\xi \wedge \overline{\Omega}_\xi = 8i \operatorname{vol}(X_\xi)$, i.e.

$$\begin{aligned} \frac{1}{2i} \Omega_\xi \wedge \overline{\Omega}_\xi = \operatorname{Im} \Omega_\xi \wedge \operatorname{Re} \Omega_\xi &= (\xi \lrcorner * \varphi) \wedge [\xi \lrcorner (\xi^\# \wedge \varphi)] \\ &= -\xi \lrcorner [(\xi \lrcorner * \varphi) \wedge (\xi^\# \wedge \varphi)] \\ &= \xi \lrcorner [*(\xi^\# \wedge \varphi) \wedge (\xi^\# \wedge \varphi)] && \text{by (10)} \\ &= |\xi^\# \wedge \varphi|^2 \xi \lrcorner \operatorname{vol}(M) \\ &= 4|\xi^\#|^2 (*\xi^\#) = 4 \operatorname{vol}(X_\xi). && \text{by (7)} \end{aligned}$$

\square

We can easily calculate $*\operatorname{Re} \Omega_\xi = -\operatorname{Im} \Omega_\xi \wedge \xi^\#$ and $*\operatorname{Im} \Omega_\xi = \operatorname{Re} \Omega_\xi \wedge \xi^\#$. In particular if \star is the star operator of X_ξ (for example by (15) $\star \omega_\xi = \omega_\xi^2/2$), then

$$(16) \quad \star \operatorname{Re} \Omega_\xi = \operatorname{Im} \Omega_\xi.$$

Notice that ω_ξ is a symplectic structure on X_ξ whenever $d\varphi = 0$ and $\mathcal{L}_\xi(\varphi)|_{V_\xi} = 0$, where \mathcal{L}_ξ denotes the Lie derivative along ξ . This is because $\omega_\xi = \xi \lrcorner \varphi$ and:

$$d\omega_\xi = \mathcal{L}_\xi(\varphi) - \xi \lrcorner d\varphi = \mathcal{L}_\xi(\varphi).$$

Also $d^*\varphi = 0 \implies d^*\omega_\xi = 0$, without any condition on the vector field ξ , since

$$(17) \quad *\varphi = *\omega_\xi - \operatorname{Im} \Omega_\xi \wedge \xi^\#,$$

and hence $d(\star\omega_\xi) = d(\star\varphi|_{X_\xi}) = 0$. Also $d\varphi = 0 \implies d(\text{Re } \Omega_\xi) = d(\varphi|_{X_\xi}) = 0$.

Furthermore, $d^*\varphi = 0$ and $\mathcal{L}_\xi(\star\varphi)|_{V_\xi} = 0 \implies d(\text{Im } \Omega_\xi) = 0$; this is because $\text{Im } \Omega_\xi = \xi \lrcorner (\star\varphi)$, where \star is the star operator on (M, φ) . Also, J_ξ is integrable when $d\Omega = 0$ (e.g. [Hi]). By using the following definition, we can sum up all the conclusions of the above discussion as Theorem 5 below.

Definition 6. (X^6, ω, Ω, J) is called an almost Calabi-Yau manifold, if X is a Riemannian manifold with a non-degenerate 2-form ω (i.e. $\omega^3 = 6\text{vol}(X)$) which is co-closed, and J is a metric invariant almost complex structure which is compatible with ω , and Ω is a non-vanishing $(3, 0)$ form with $\text{Re } \Omega$ closed. Furthermore, when ω and $\text{Im } \Omega$ are closed, we call this a Calabi-Yau manifold.

Theorem 5. Let (M, φ) be a G_2 manifold, and ξ be a unit vector field which comes from a codimension one foliation on M , then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is an almost Calabi-Yau manifold with $\varphi|_{X_\xi} = \text{Re } \Omega_\xi$ and $\star\varphi|_{X_\xi} = \star\omega_\xi$. Furthermore, if $\mathcal{L}_\xi(\varphi)|_{X_\xi} = 0$ then $d\omega_\xi = 0$, and if $\mathcal{L}_\xi(\star\varphi)|_{X_\xi} = 0$ then J_ξ is integrable; when both of these conditions are satisfied then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is a Calabi-Yau manifold.

Remark 4. If ξ and ξ' are sections of \mathbf{V} and \mathbf{E} respectively, then from [M] the condition $\mathcal{L}_\xi(\star\varphi)|_{X_\xi} = 0$ (complex geometry of X_ξ) implies that deforming associative submanifolds of X_ξ along ξ in M keeps them associative; and $\mathcal{L}_{\xi'}(\varphi)|_{X_{\xi'}} = 0$ (symplectic geometry of $X_{\xi'}$) implies that deforming coassociative submanifolds of $X_{\xi'}$ along ξ' in M keeps them coassociative (e.g. for an example see Example 1).

Notice that both complex and symplectic structure of the CY-manifold X_ξ in Theorem 3 is determined by φ when they exist. Recall that (c.f. [V]) elements $\Omega \in H^{3,0}(X_\xi, \mathbb{C})$ along with topology of X_ξ (i.e. the intersection form of $H^3(X_\xi, \mathbb{Z})$) parametrize complex structures on X_ξ as follows: We compute the third betti number $b_3(M) = 2h^{2,1} + 2$ since

$$H^3(X_\xi, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} = 2(\mathbb{C} \oplus H^{2,1}).$$

Let $\{A^i, B_j\}$ be a symplectic basis of $H_3(X, \mathbb{Z})$, $i = 1, \dots, h^{2,1} + 1$, then

$$(18) \quad X_i = \int_{A^i} \Omega$$

give complex numbers which are local homeogenous coordinates of the moduli space of complex structures on X_ξ , which is a $h^{2,1}$ dimensional space (there is an extra parameter here since Ω is defined up to scale multiplication).

As we have seen in the example of the Section 3.1, the choice of ξ can give rise to quite different complex structures on X_ξ (e.g. $SU(2)$ and $SU(3)$ structures). For example, assume $\xi \in \Omega^0(M, \mathbf{V})$ and $\xi' \in \Omega^0(M, \mathbf{E})$ be unit vector fields, such that the the codimension one plane fields ξ^\perp and ξ'^\perp come from foliations. Let X_ξ and $X_{\xi'}$ be pages of the corresponding foliations. By our definition X_ξ and $X_{\xi'}$ are

mirror duals of each other. Decomposition $T(M) = \mathbf{E} \oplus \mathbf{V}$ gives rise to splittings $TX_\xi = \mathbf{E} \oplus \bar{\mathbf{E}}$, and $TX_{\xi'} = \mathbf{C} \oplus \mathbf{V}$, where $\bar{\mathbf{E}} = \xi^\perp(\mathbf{V}) \subset \mathbf{V}$ is a 3-dimensional subbundle, and $\mathbf{C} = (\xi')^\perp(\mathbf{E}) \subset \mathbf{E}$ is a 2-dimensional subbundle. Furthermore, \mathbf{E} is Lagrangian in TX_ξ i.e. $J_\xi(\mathbf{E}) = \bar{\mathbf{E}}$, and \mathbf{C}, \mathbf{V} are complex in $TX_{\xi'}$ i.e. $J_{\xi'}(\mathbf{C}) = \mathbf{C}$ and $J_{\xi'}(\mathbf{V}) = \mathbf{V}$. Also notice that, $Re \Omega_\xi$ is a calibration form of \mathbf{E} , and ω_ξ is a calibration form of \mathbf{C} . In particular, $\langle \Omega_\xi, \mathbf{E} \rangle = 1$ and $\langle \omega_\xi \wedge \xi^\#, \mathbf{E} \rangle = 0$; and $\langle \Omega_{\xi'}, \mathbf{E} \rangle = 0$ and $\langle \omega_{\xi'} \wedge (\xi')^\#, \mathbf{E} \rangle = 1$.

If X_ξ and $X_{\xi'}$ are strong duals of each other, we can find a homotopy of non-vanishing unit vector fields ξ_t ($0 \leq t \leq 1$) starting with $\xi \in \mathbf{V}$ ending with $\xi' \in \mathbf{E}$. This gives a 7-plane distribution $\Xi = \xi_t^\perp \oplus \frac{\partial}{\partial t}$ on $M \times [0, 1]$ with integral submanifolds $X_\xi \times [0, \epsilon)$ and $X_{\xi'} \times (1 - \epsilon, 1]$ on a neighborhood of the boundary. Then by [Th] and [Th1] we can homotop Ξ to a foliation extending the foliation on the boundary (possibly by taking ϵ smaller). Let $Q^7 \subset M \times [0, 1]$ be the smooth manifold given by this foliation, with $\partial Q = X_\xi \cup X_{\xi'}$, where $X_\xi \subset M \times \{0\}$ and $X_{\xi'} \subset M \times \{1\}$.

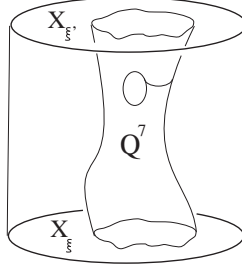


FIGURE 2.

We can define $\Phi \in \Omega^3(M \times [0, 1])$ with $\Phi|_{X_\xi} = \Omega_\xi$ and $\Phi|_{X_{\xi'}} = \xi \lrcorner \star \omega_{\xi'}$

$$\Phi = \Phi(\varphi, \Lambda, t) = \langle \omega_{\xi_t} \wedge \xi_t^\#, \mathbf{E} \rangle \xi_t'' \lrcorner \star \omega_{\xi_t} + \langle Re \Omega_{\xi_t}, \mathbf{E} \rangle \Omega_{\xi_t}$$

where $\xi_t'' = J_{\xi \times \xi'}(\xi_t) = \xi_t \times (\xi \times \xi')$ (hence $\xi_0'' = -\xi'$ and $\xi_1'' = \xi$). This can be viewed as a correspondence between the complex structure of X_ξ and the symplectic structure of $X_{\xi'}$. In general, the manifold pairs X_α and X_β (as constructed in Theorem 5) determine each others almost Calabi -Yau structures via φ provided they are defined.

Proposition 6. *Let $\{\alpha, \beta\}$ be orthonormal vector fields on (M, φ) . Then on X_α the following hold*

$$(i) \quad Re \Omega_\alpha = \omega_\beta \wedge \beta^\# + Re \Omega_\beta$$

$$(ii) \quad Im \, \Omega_\alpha = \alpha \lrcorner (\star \omega_\beta) - (\alpha \lrcorner Im \, \Omega_\beta) \wedge \beta^\#$$

$$(iii) \quad \omega_\alpha = \alpha \lrcorner Re \, \Omega_\beta + (\alpha \lrcorner \omega_\beta) \wedge \beta^\#$$

Proof. Since $Re \, \Omega_\alpha = \varphi|_{X_\alpha}$ (i) follows. Since $Im \, \Omega_\alpha = \alpha \lrcorner \star \varphi$ following gives (ii)

$$\begin{aligned} \alpha \lrcorner (\star \omega_\beta) &= \alpha \lrcorner [\beta \lrcorner (\beta \lrcorner \varphi)] \\ &= \alpha \lrcorner \beta \lrcorner (\beta^\# \wedge \star \varphi) \\ &= \alpha \lrcorner \star \varphi + \beta^\# \wedge (\alpha \lrcorner \beta \lrcorner \star \varphi) \\ &= \alpha \lrcorner \star \varphi + (\alpha \lrcorner Im \, \Omega_\beta) \wedge \beta^\# \end{aligned}$$

(iii) follows from the following computation

$$\alpha \lrcorner Re \, \Omega_\beta = \alpha \lrcorner \beta \lrcorner (\beta^\# \wedge \varphi) = \alpha \lrcorner \varphi + \beta^\# \wedge (\alpha \lrcorner \beta \lrcorner \varphi) = \alpha \lrcorner \varphi - (\alpha \lrcorner \omega_\beta) \wedge \beta^\#$$

□

Notice that even though the identities of Proposition 6 hold only after restricting the right hand side to X_α , all the individual terms are defined everywhere on (M, φ) . Also, from the construction, X_α and X_β inherit vector fields β and α , respectively.

Corollary 7. *Let $\{\alpha, \beta\}$ be orthonormal vector fields on (M, φ) . Then there are $A_{\alpha\beta} \in \Omega^3(M)$, and $W_{\alpha\beta} \in \Omega^2(M)$ satisfying*

$$\begin{aligned} (a) \quad & \varphi|_{X_\alpha} = Re \, \Omega_\alpha \quad \text{and} \quad \varphi|_{X_\beta} = Re \, \Omega_\beta \\ (b) \quad & A_{\alpha\beta}|_{X_\alpha} = Im \, \Omega_\alpha \quad \text{and} \quad A_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner (\star \omega_\beta) \\ (c) \quad & W_{\alpha\beta}|_{X_\alpha} = \omega_\alpha \quad \text{and} \quad W_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner Re \, \Omega_\beta \end{aligned}$$

For example, when φ varies through metric preserving G_2 structures [B2], (hence fixing the orthogonal frame $\{\xi, \xi'\}$), it induces variations of ω one side, and Ω on the other side.

Remark 5. *By using Proposition 6, in the previous torus example of 3.1 one can show a natural correspondence between the groups $H^{2,1}(X_\xi)$ and $H^{1,1}(X_{\xi'})$. Even though \mathbb{T}^7 is a trivial example of a G_2 manifold, it is an important special case since the G_2 manifolds of Joyce are obtained by smoothing quotients of \mathbb{T}^7 by finite group actions. We believe this process turns the subtori X_ξ 's into Borcea-Voisin manifolds with a similar correspondence of their cohomology groups.*

For the discussion of the previous paragraph to work, we need a non-vanishing vector field ξ in $T(M) = \mathbf{E} \oplus \mathbf{V}$, moving from \mathbf{V} to \mathbf{E} . The bundle \mathbf{E} always has a non-zero section, in fact it has a non-vanishing orthonormal 3-frame field; but \mathbf{V} may not have a non-zero section. Nevertheless the bundle $\mathbf{V} \rightarrow M$ does have a non-vanishing section in the complement of a 3-manifold $Y \subset M$, which is a transverse self intersection of the zero section. In [AS], Seiberg-Witten equations

of such 3-manifolds were related to associative deformations. So we can use these partial sections and, as a consequence X_ξ and $X_{\xi'}$ may not be closed manifolds. The following is a useful example:

Example 1. Let X_1, X_2 be two Calabi-Yau manifolds, where X_1 is the cotangent bundle of S^3 and X_2 is the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle of S^2 . They are conjectured to be the mirror duals of each other by physicists (c.f. [Ma]). By using the approach of this paper, we identify them as 6-dimensional submanifolds of a G_2 manifold. Let's choose $M = \wedge_+^2(S^4)$; this is a G_2 manifold by Bryant-Salamon [BS].

Let $\pi : \wedge_+^2(S^4) \rightarrow S^4$ be the bundle projection. The sphere bundle of π (which is also $\overline{\mathbb{CP}^3}$) is the so-called twistor bundle, let us denote it by $\pi_1 : Z(S^4) \rightarrow S^4$. It is known that the normal bundle of each fiber $\pi_1^{-1}(p) \cong S^2$ in $Z(S^4)$ can be identified by $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ [S]. Now we take \mathbf{E} to be the bundle of vertical tangent vectors of π , and $\mathbf{V} = \pi^*(TS^4)$, lifted by connection distribution. Let ξ be the pull-back of the vector field on S^4 with two zeros (flowing from north pole \mathbf{n} to south pole \mathbf{s}), and let ξ' be the radial vector field of \mathbf{E} . Clearly $X_\xi = T^*(S^3)$ and $X_{\xi'} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Note that ξ is non-vanishing in the complement of $\pi^{-1}\{\mathbf{n}, \mathbf{s}\}$, whereas ξ' is non-vanishing in the complement of the zero section of π . Clearly on the set where they are both defined, ξ and ξ' are homotopic through nonvanishing vector fields ξ_t . This would define a cobordism between the complements of the zero sections of the bundles $T^*(S^3)$ and $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, if the distributions ξ_t^\perp were involutive.

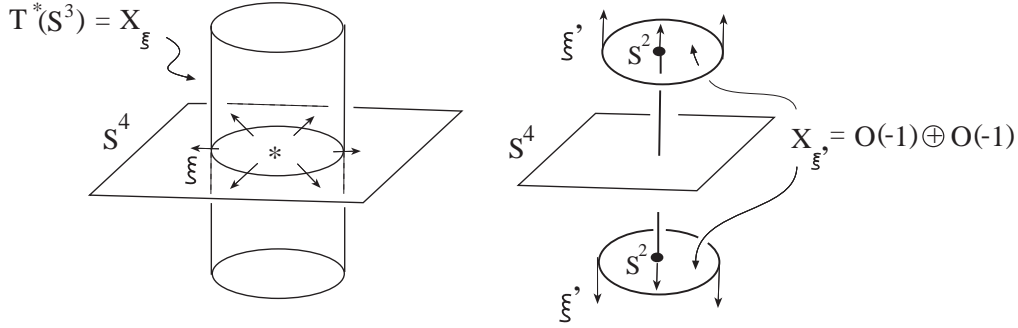


FIGURE 3.

Here the change of complex structures $X_{\xi'} \rightsquigarrow X_\xi$ happens as follows. Let $S_\lambda^3 \rightarrow S^2$ be the Hopf map with fibers consisting of circles of radius λ , clearly $S_\infty^3 = S^2 \times \mathbb{R}$

$$(\mathbb{C}^2 - 0) \times S^2 \rightarrow (S_\lambda^3 \times \mathbb{R}) \times S^2 \xrightarrow{\lambda \rightarrow \infty} S_\infty^3 \times S_\infty^3$$

where the complex structure on $S_\infty^3 \times S_\infty^3$ is the obvious one, induced from exchanging the factors. In general if we allow the vector fields ξ and ξ' be homotopic through

vector fields ξ_t possibly with zeros, or the family ξ_t^\perp not remain involutive the cobordism between X_ξ and $X_{\xi'}$ will have singularities.

Remark 6. *If we apply the construction of Example 1 to the total space of the spinor bundle $Q \rightarrow S^3$ (see Remark 2), the two dual 6-manifolds we get are $S^2 \times \mathbb{R}^4$ and $S^3 \times \mathbb{R}^3$.*

There is also a concept of mirror-dual G_2 manifolds in a $\text{Spin}(7)$ manifold, hence we can talk about mirror dual CY manifolds coming from two different mirror dual G_2 submanifolds of a $\text{Spin}(7)$ manifold. This is the the subject of the next section.

4. MIRROR DUALITY IN $\text{Spin}(7)$ MANIFOLDS

Similar to Calabi-Yau case there is a notion of mirror duality between G_2 manifolds [Ac], [AV], [GYZ], [SV]. In this section we will give a definition of mirror G_2 pairs, and an example which shows that associative and co-associative geometries in mirror G_2 pairs are induced from the same calibration 4-form in a $\text{Spin}(7)$ manifold, and hence these geometries dual to each other. Let us first recall the basic definitions and properties of $\text{Spin}(7)$ geometries. The main references in this subject are [HL] and [Ti].

Definition 7. *An 8-dimensional Riemannian manifold (N, Ψ) is called a $\text{Spin}(7)$ manifold if the holonomy group of its metric connection lies in $\text{Spin}(7) \subset GL(8)$.*

Equivalently, a $\text{Spin}(7)$ manifold is an 8-dimensional Riemannian manifold with a triple cross product \times on its tangent bundle, and a closed 4-form $\Psi \in \Omega^4(N)$ with

$$\Psi(u, v, w, z) = \langle u \times v \times w, z \rangle.$$

Definition 8. *A 4-dimensional submanifold X of a $\text{Spin}(7)$ manifold (N, Ψ) is called Cayley if $\Psi|_X \equiv \text{vol}(X)$.*

Analogous to the G_2 case, we introduce a tangent bundle valued 3-form, which is just the triple cross product of N .

Definition 9. *Let (N, Ψ) be a $\text{Spin}(7)$ manifold. Then $\Upsilon \in \Omega^3(N, TN)$ is the tangent bundle valued 3-form defined by the identity:*

$$\langle \Upsilon(u, v, w), z \rangle = \Psi(u, v, w, z) = \langle u \times v \times w, z \rangle.$$

$\text{Spin}(7)$ manifolds can be constructed from G_2 manifolds. Let (M, φ) be a G_2 manifold with a 3-form φ , then $M \times S^1$ (or $M \times \mathbb{R}$) has holonomy group $G_2 \subset \text{Spin}(7)$, hence is a $\text{Spin}(7)$ manifold. In this case $\Psi = \varphi \wedge dt + *_7 \varphi$, where $*_7$ is the star operator of M^7 .

Now we will repeat a similar construction for a $\text{Spin}(7)$ manifold (N, Ψ) , which we did for G_2 manifolds. Here we make an assumption that $T(M)$ admits a non-vanishing 3-frame field $\Lambda = \langle u, v, w \rangle$, then we decompose $T(M) = \mathbf{K} \oplus \mathbf{D}$, where

$\mathbf{K} = \langle u, v, w, u \times v \times w \rangle$ is the bundle of Cayley 4-planes (where Ψ restricts to be 1) and \mathbf{D} is the complementary subbundle (note that this is also a bundle of Cayley 4-planes since the form Ψ is self dual). In the G_2 case, existence of an analogous decomposition of the tangent bundle followed from [T] (in this case we can just restrict to a submanifold which a 3-frame field exists). On a chart in N let e_1, \dots, e_8 be an orthonormal frame and e^1, \dots, e^8 be the dual coframe, then the calibration 4-form is given as (c.f. [HL])

$$(19) \quad \begin{aligned} \Psi = & e^{1234} + (e^{12} - e^{34}) \wedge (e^{56} - e^{78}) \\ & + (e^{13} + e^{24}) \wedge (e^{57} + e^{68}) \\ & + (e^{14} - e^{23}) \wedge (e^{58} - e^{67}) + e^{5678} \end{aligned}$$

which is a self dual 4-form, and the corresponding tangent bundle valued 3-form is

$$\begin{aligned} \Upsilon = & (e^{234} + e^{256} - e^{278} + e^{357} + e^{368} + e^{458} - e^{467})e_1 \\ & + (-e^{134} - e^{156} + e^{178} + e^{457} + e^{468} - e^{358} + e^{367})e_2 \\ & + (e^{124} - e^{456} + e^{478} - e^{157} - e^{168} + e^{258} - e^{267})e_3 \\ & + (-e^{123} + e^{356} - e^{378} - e^{257} - e^{268} - e^{158} + e^{167})e_4 \\ & + (e^{126} - e^{346} + e^{137} + e^{247} + e^{148} - e^{238} + e^{678})e_5 \\ & + (-e^{125} + e^{345} + e^{138} + e^{248} - e^{147} + e^{237} - e^{578})e_6 \\ & + (-e^{128} + e^{348} - e^{135} - e^{245} + e^{146} - e^{236} + e^{568})e_7 \\ & + (e^{127} - e^{347} - e^{136} - e^{246} - e^{145} + e^{235} - e^{567})e_8. \end{aligned}$$

This time we show that the form Υ induce G_2 structures on certain subbundles of $T(N)$. Let γ be a nowhere vanishing vector field of N . We define a G_2 structure φ_γ on the 7-plane bundle $V_\gamma := \gamma^\perp$ by (where $*_8$ is the star operator on N^8)

$$(20) \quad \varphi_\gamma := \langle \Upsilon, \gamma \rangle = \gamma \lrcorner \Psi = *_8(\Psi \wedge \gamma^\#).$$

Assuming that V_γ comes from a foliation, we let M_γ be an integral submanifold of V_γ . We have $d\varphi_\gamma = 0$, provided $\mathcal{L}_\gamma(\Psi)|_{V_\gamma} = 0$. On the other hand, we always have $d(\star\varphi_\gamma) = 0$ on M_γ . To see this, we use

$$\Psi = \varphi_\gamma \wedge \gamma^\# + *_7\varphi_\gamma$$

where $*_7$ is the star operator on M_γ , and use $d\Psi = 0$ and the foliation condition $d\gamma^\# \wedge \gamma^\# = 0$, and the identity $\theta|_{M_\gamma} = \gamma \lrcorner [\theta \wedge \gamma^\#]$ for forms θ . In order to state the next theorem, we need a definition:

Definition 10. *A manifold with G_2 structure (M, φ) is called an almost G_2 -manifold if φ is co-closed.*

Theorem 8. *Let (N^8, Ψ) be a $\text{Spin}(7)$ manifold, and γ be a unit vector field which comes from a foliation, then $(M_\gamma, \varphi_\gamma)$ is an almost G_2 manifold. Furthermore if $\mathcal{L}_\gamma(\Psi)|_{M_\gamma} = 0$ then $(M_\gamma, \varphi_\gamma)$ is a G_2 manifold.*

Proof. Follows by checking Definition 1 and by the discussion above. \square

The following theorem says the induced G_2 structures on M_α, M_β determine each other via Ψ ; more specifically φ_α and φ_β are restrictions of a global 3-form of N .

Proposition 9. *Let (N, Ψ) be a $\text{Spin}(7)$ manifold, and $\{\alpha, \beta\}$ be an orthonormal vector fields on N . Then the following holds on M_α*

$$\varphi_\alpha = -\alpha \lrcorner (\varphi_\beta \wedge \beta^\# + *_7 \varphi_\beta)$$

Proof. The proof follows from the definitions, and by expressing φ_α and φ_β in terms of $\beta^\#$ and $\alpha^\#$ by the formula (13). \square

As in the G_2 case, by choosing different γ 's, one can find various different G_2 manifolds M_γ with interesting structures. Most interestingly, we will get certain “dual” M_γ 's by choosing γ in \mathbf{K} or in \mathbf{D} . This will shed light on more general version of mirror symmetry of Calabi-Yau manifolds. First we will discuss an example.

4.1. An example.

Let $\mathbb{T}^8 = \mathbb{T}^4 \times \mathbb{T}^4$ be the $\text{Spin}(7)$ 8-torus, where $\{e_1, e_2, e_3, e_4\}$ is the basis for the Cayley \mathbb{T}^4 and $\{e_5, e_6, e_7, e_8\}$ is the basis for the complementary \mathbb{T}^4 . We can take the corresponding calibration 4-form (20) above, and take the decomposition $T(N) = \mathbf{K} \oplus \mathbf{D}$, where $\{e_1, e_2, e_3, e_4\}$ is the orthonormal basis for the Cayley bundle \mathbf{K} , and $\{e_5, e_6, e_7, e_8\}$ is the local orthonormal basis for the complementary bundle \mathbf{D} . Then if we choose $\gamma = e_4 = e_1 \times e_2 \times e_3$ then we get

$$\varphi_\gamma = -e^{123} + e^{356} - e^{378} - e^{257} - e^{268} - e^{158} + e^{167}$$

On the other hand, if we choose $\gamma' = e_5$ then we get

$$\varphi_{\gamma'} = e^{126} - e^{346} + e^{137} + e^{247} + e^{148} - e^{238} + e^{678}$$

which give different G_2 structures on the 7 tori M_γ and $M_{\gamma'}$.

Note that if we choose γ from the Cayley bundle \mathbf{K} , we get the G_2 structure on the 7-torus M_γ which reduces the Cayley 4-torus $\mathbb{T}^4 = \mathbb{T}^3 \times S^1$ (where γ is tangent to S^1 direction) to an associative 3-torus $\mathbb{T}^3 \subset M_\gamma$ with respect to this G_2 structure. On the other hand if we choose γ' from the complementary bundle \mathbf{D} , then the Cayley 4-torus \mathbb{T}^4 will be a coassociative submanifold of the 7-torus $M_{\gamma'}$ with the corresponding G_2 structure. Hence associative and coassociative geometries are dual to each other as they are induced from the same calibration 4-form Ψ on a

$\text{Spin}(7)$ manifold. This suggests the following definition of the “mirror duality” for G_2 manifolds.

Definition 11. *Two 7-manifolds with G_2 structures are mirror pairs, if their G_2 -structures are induced from the same calibration 4-form in a $\text{Spin}(7)$ manifold. Furthermore they are strong duals if their normal vector fields are homotopic.*

Remark 7. *For example, by [BS] the total space of an \mathbb{R}^4 bundle over S^4 has a $\text{Spin}(7)$ structure. By applying the process of Example 1, we obtain mirror pairs M_γ and $M_{\gamma'}$ to be $S^3 \times \mathbb{R}^4$ and $\mathbb{R}^4 \times S^3$ with dual G_2 structures.*

4.2. Dual Calabi-Yau’s inside of $\text{Spin}(7)$.

Let (N^8, Ψ) be a $\text{Spin}(7)$ manifold, and let $\{\alpha, \beta\}$ be an orthonormal 2-frame field in N , each coming from a foliation. Let $(M_\alpha, \varphi_\alpha)$ and (M_β, φ_β) be the G_2 manifolds given by Theorem 8. Similarly by Theorem 5, the vector fields β in M_α , and α in M_β give almost Calabi-Yau’s $X_{\alpha\beta} \subset M_\alpha$ and $X_{\beta\alpha} \subset M_\beta$. Let us denote $X_{\alpha\beta} = (X_{\alpha\beta}, \omega_{\alpha\beta}, \Omega_{\alpha\beta}, J_{\alpha\beta})$ likewise $X_{\beta\alpha} = (X_{\beta\alpha}, \omega_{\beta\alpha}, \Omega_{\beta\alpha}, J_{\beta\alpha})$. Then we have

Proposition 10. *The following relations hold:*

- (i) $J_{\alpha\beta}(u) = u \times \beta \times \alpha$
- (ii) $\omega_{\alpha\beta} = \beta \lrcorner \alpha \lrcorner \Psi$
- (iii) $\text{Re } \Omega_{\alpha\beta} = \alpha \lrcorner \Psi|_{X_{\alpha\beta}}$
- (iv) $\text{Im } \Omega_{\alpha\beta} = \beta \lrcorner \Psi|_{X_{\alpha\beta}}$

Proof. (i), (ii), and (iii) follow from definitions, and the formula $X \times Y = (X \lrcorner Y \lrcorner \varphi)^\#$.

$$\begin{aligned}
 \text{Im } \Omega_{\alpha\beta} &= \beta \lrcorner *_7 \varphi_\alpha = \beta \lrcorner *_7 (\alpha \lrcorner \Psi) \\
 &= \beta \lrcorner [\alpha \lrcorner *_8 (\alpha \lrcorner \Psi)] \\
 &= \beta \lrcorner [\alpha \lrcorner (\alpha^\# \wedge \Psi)] \quad \text{by (10)} \\
 &= \beta \lrcorner [\Psi - \alpha^\# \wedge (\alpha \lrcorner \Psi)] \\
 &= \beta \lrcorner \Psi - \alpha^\# \wedge (\beta \lrcorner \alpha \lrcorner \Psi).
 \end{aligned}$$

Left hand side is already defined on $X_{\alpha\beta}$, by restricting to $X_{\alpha\beta}$ we get (iii). \square

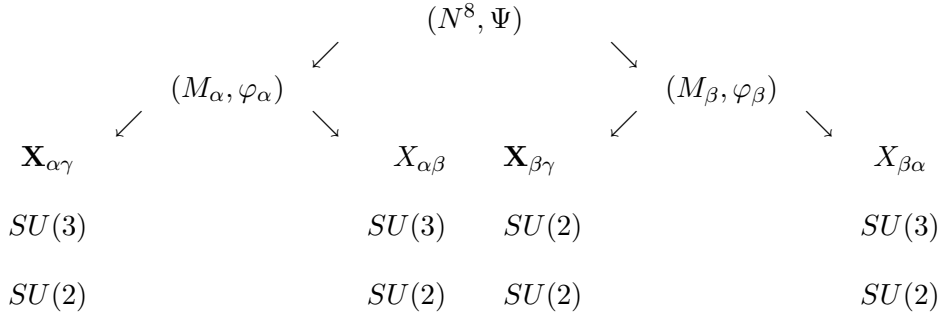
Corollary 11. *When $X_{\alpha,\beta}$ and $X_{\beta,\alpha}$ coincide, they are oppositely oriented manifolds and $\omega_{\alpha,\beta} = -\omega_{\beta,\alpha}$, and $\text{Re } \Omega_{\alpha\beta} = -\text{Im } \Omega_{\beta\alpha}$ (as forms on $X_{\alpha\beta}$).*

Now let $\{\alpha, \beta, \gamma\}$ be an orthonormal 3-frame field in (N^8, Ψ) , and let $(M_\alpha, \varphi_\alpha)$, (M_β, φ_β) , and $(M_\gamma, \varphi_\gamma)$ be the corresponding almost G_2 manifolds. As before, the

orthonormal vector fields $\{\gamma, \beta\}$ in M_α and $\{\gamma, \alpha\}$ in M_β give rise to corresponding almost Calabi-Yau's $X_{\alpha, \gamma}$, $X_{\alpha, \beta}$ in M_α , and $X_{\beta, \gamma}$, $X_{\beta, \alpha}$ in M_β .

In this way (N^8, Ψ) gives rise to 4 Calabi-Yau descendents. By Corollary 11, $X_{\alpha\beta}$ and $X_{\beta\alpha}$ are different geometrically; they may not even be the same as smooth manifolds, but for simplicity we may consider it to be the same smooth manifold obtained from the triple intersection of the three G_2 manifolds.

In case we have a decomposition $T(N) = \mathbf{K} \oplus \mathbf{D}$ of the tangent bundle of (N^8, Ψ) by Cayley plus its orthogonal bundles (Section 4); we can choose our frame special and obtain interesting CY-manifolds. For example, if we choose $\alpha \in \Omega^0(M, \mathbf{K})$ and $\beta, \gamma \in \Omega^0(N, \mathbf{D})$ we get one set of complex structures, whose types are indicated by the first row of the following diagram. On the other hand, if we choose all $\{\alpha, \beta, \gamma\}$ lie entirely in \mathbf{K} or \mathbf{D} we get an other set of complex structures, as indicated by the second row of the diagram.



Here all the corresponding symplectic and the holomorphic forms of the resulting Calabi-Yau's come from restriction of global forms induced by Ψ . The following gives relations between the complex/symplectic structures of these induced CY-manifolds; i.e. the structures $X_{\alpha\gamma}$, $X_{\beta\gamma}$ and $X_{\alpha\beta}$ satisfy a certain triality relation.

Proposition 12. *We have the following relations;*

- (i) $Re \Omega_{\alpha\gamma} = \alpha \lrcorner (*_6 \omega_{\beta\gamma}) + \omega_{\alpha\beta} \wedge \beta^\#$
- (ii) $Im \Omega_{\alpha\gamma} = \omega_{\beta\gamma} \wedge \beta^\# - \gamma \lrcorner *_6 (\omega_{\alpha\beta})$
- (iii) $\omega_{\alpha\gamma} = \alpha \lrcorner Im \Omega_{\beta\gamma} + (\gamma \lrcorner \omega_{\alpha\beta}) \wedge \beta^\#$

First we need to prove a lemma;

Lemma 13. *The following relations hold;*

$$\begin{aligned}
 \alpha \lrcorner *_6 (\omega_{\beta\gamma}) &= \alpha \lrcorner \Psi + \gamma^\# \wedge (\alpha \lrcorner \gamma \lrcorner \Psi) + \beta^\# \wedge (\alpha \lrcorner \beta \lrcorner \Psi) - \gamma^\# \wedge \beta^\# \wedge (\alpha \lrcorner \gamma \lrcorner \beta \lrcorner \Psi). \\
 Im \Omega_{\beta\gamma} &= -\gamma \lrcorner \Psi - \beta^\# \wedge (\gamma \lrcorner \beta \lrcorner \Psi). \\
 Re \Omega_{\alpha\gamma} &= \alpha \lrcorner \Psi - \gamma^\# \wedge (\gamma \lrcorner \alpha \lrcorner \Psi).
 \end{aligned}$$

Proof.

$$\begin{aligned}
\alpha \lrcorner *6 (\omega_{\beta\gamma}) &= \alpha \lrcorner [\gamma \lrcorner \beta \lrcorner *8 (\gamma \lrcorner \beta \lrcorner \Psi)] \\
&= -\alpha \lrcorner \gamma \lrcorner \beta \lrcorner (\gamma^\# \wedge \beta^\# \wedge \Psi) \\
&= -\alpha \lrcorner \gamma \lrcorner [-\gamma^\# \wedge \Psi + \gamma^\# \wedge \beta^\# \wedge (\beta \lrcorner \Psi)] \\
&= \alpha \lrcorner \Psi + \gamma^\# \wedge (\alpha \lrcorner \gamma \lrcorner \Psi) + \beta^\# \wedge (\alpha \lrcorner \beta \lrcorner \Psi) \\
&\quad - \gamma^\# \wedge \beta^\# \wedge (\alpha \lrcorner \gamma \lrcorner \beta \lrcorner \Psi).
\end{aligned}$$

$$\begin{aligned}
Im \Omega_{\beta\gamma} &= \gamma \lrcorner *7 (\beta \lrcorner \Psi) \\
&= \gamma \lrcorner [\beta \lrcorner *8 (\beta \lrcorner \Psi)] \\
&= -\gamma \lrcorner \beta \lrcorner (\beta^\# \wedge \Psi) \\
&= -\gamma \lrcorner \Psi - \beta^\# \wedge (\gamma \lrcorner \beta \lrcorner \Psi).
\end{aligned}$$

$$Re \Omega_{\alpha\gamma} = (\alpha \lrcorner \Psi)|_{X_{\alpha\gamma}} = \gamma \lrcorner [\gamma^\# \wedge (\alpha \lrcorner \Psi)] = \alpha \lrcorner \Psi - \gamma^\# \wedge (\gamma \lrcorner \alpha \lrcorner \Psi).$$

□

Proof of Proposition 12. We calculate the following by using Lemma 13:

$$\begin{aligned}
\alpha \lrcorner *6 (\omega_{\beta\gamma}) + \omega_{\alpha\beta} \wedge \beta^\# &= \alpha \lrcorner \Psi + \gamma^\# \wedge (\alpha \lrcorner \gamma \lrcorner \Psi) + \beta^\# \wedge (\alpha \lrcorner \beta \lrcorner \Psi) \\
&\quad - \gamma^\# \wedge \beta^\# \wedge (\alpha \lrcorner \gamma \lrcorner \beta \lrcorner \Psi) + (\beta \lrcorner \alpha \lrcorner \Psi) \wedge \beta^\# \\
&= \alpha \lrcorner \Psi - \gamma^\# \wedge (\gamma \lrcorner \alpha \lrcorner \Psi) - \gamma^\# \wedge \beta^\# \wedge (\alpha \lrcorner \gamma \lrcorner \beta \lrcorner \Psi).
\end{aligned}$$

Since we are restricting to $X_{\alpha\gamma}$ we can throw away terms containing $\gamma^\#$ and get (i). We prove (ii) similarly:

$$\begin{aligned}
Im \Omega_{\alpha\gamma} &= (\gamma \lrcorner *7 \varphi_\alpha) = \gamma \lrcorner [\alpha \lrcorner *8 \varphi_\alpha] \\
&= \gamma \lrcorner [\alpha \lrcorner *8 (\alpha \lrcorner \Psi)] = -\gamma \lrcorner [\alpha \lrcorner (\alpha^\# \wedge \Psi)] \\
&= -\gamma \lrcorner \Psi + \gamma \lrcorner [\alpha^\# \wedge (\alpha \lrcorner \Psi)] \\
&= -\gamma \lrcorner \Psi - \alpha^\# \wedge (\gamma \lrcorner \alpha \lrcorner \Psi).
\end{aligned}$$

$$\begin{aligned}
\omega_{\beta\gamma} \wedge \beta^\# - \gamma \lrcorner (*6 \omega_{\alpha\beta}) &= -\gamma \lrcorner (*6 \omega_{\alpha\beta}) + (\gamma \lrcorner \beta \lrcorner \Psi) \wedge \beta^\# \\
&= -\gamma \lrcorner \Psi - \beta^\# \wedge (\gamma \lrcorner \beta \lrcorner \Psi) - \alpha^\# \wedge (\gamma \lrcorner \alpha \lrcorner \Psi) \\
&\quad + \beta^\# \wedge \alpha^\# \wedge (\gamma \lrcorner \beta \lrcorner \alpha \lrcorner \Psi) + (\gamma \lrcorner \beta \lrcorner \Psi) \wedge \beta^\#.
\end{aligned}$$

Here, we used Lemma 13 with different indices $(\alpha, \beta, \gamma) \mapsto (\gamma, \alpha, \beta)$, and since we are restricting to $X_{\alpha\gamma}$ we threw away terms containing $\alpha^\#$. Finally, (iii) follows by plugging in Lemma 13 to definitions. □

Following says that Calabi-Yau structures of $X_{\alpha\gamma}$ and $X_{\beta\gamma}$ determine each other via Ψ . Proposition 14 is basically a consequence of Proposition 6 and Corollary 11.

Proposition 14. *We have the following relations*

$$(i) \operatorname{Re} \Omega_{\alpha\gamma} = \alpha \lrcorner (*_6 \omega_{\beta\gamma}) - (\alpha \lrcorner \operatorname{Re} \Omega_{\beta\gamma}) \wedge \beta^\#$$

$$(ii) \operatorname{Im} \Omega_{\alpha\gamma} = \omega_{\beta\gamma} \wedge \beta^\# + \operatorname{Im} \Omega_{\beta\gamma}$$

$$(iii) \omega_{\alpha\gamma} = \alpha \lrcorner \operatorname{Im} \Omega_{\beta\gamma} + (\alpha \lrcorner \omega_{\beta\gamma}) \wedge \beta^\#$$

Proof. All follow from the definitions and Lemma 11 (and by ignoring $\alpha^\#$ terms). \square

Remark 8. *After this paper was written we learned that the results similar to Theorem 5 already appeared in [AW], [ASa], [C], [CS] (where they use more restricted vector fields ξ), and we also found out that the idea of studying induced hypersurface structures, from manifolds with exceptional holonomy goes back to earlier works of Calabi and Gray.*

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